Estimating the DJI Series by Multifractional Brownian Motion

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Abstract

We estimate the stock market and its price dynamics with the multifractional Brownian motion. In our analysis, we use the dataset of the Dow Jones Industrial Average (DJI) time series from March 2009 to June 2015. First, we briefly introduce the definitions and properties of the Brownian motion (Bm), fractional Brownian motion (fBm) and multifractional Brownian motion (mBm) (Ayache and Lévy Véhel, 2004). Then we model price processes as exponential of the sum of a regular process and a stochastic process and estimate the Hölder exponent. In this paper, we show how a stochastic process like mBm can be applied to simultaneously capture the fluctuations of the asset price dynamics and the long-range dependence of financial time series. Thus, we argue that with a proper functional parameter $H(t)$ we can generate mBm that can reproduce the stylized facts that characterize financial time series.

Keywords: Multifractional Brownian Motion, Hurst Parameter Estimation, Financial Times Series, Asset Price Dynamics

JEL classification: C15, C40, G00, G40

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1 Introduction

Brownian motion (Bm) and fractional Brownian motion (fBm) have been the backbone of modern finance theory and have gained huge acceptance in both academia and industry. However, the well-known Bm and fBm have certain limitations, and one of them is that the roughness of its path remains the same everywhere, i.e., the pointwise Hölder exponent of the fBm, H, is not allowed to change from one time to another (Ayache and Lévy Véhel, 2004). The multifractional Brownian motion (mBm) can overcome this. First, we introduce the Brownian motion (Bm), fractional Brownian motion (fBm) and multifractional Brownian motion (mBm). In this paper, we use the mBm as a model of financial dynamics which has the following properties: (1) locally asymptotically self-similar, (2) non-stationary Gaussian process, and (3) natural generalization of the fBm when $H(t) =$ constant. We then model the Dow Jones Industrial Average (DJI) time series and develop the algorithms to estimate the price dynamics along with the mBm. The key problem of the analysis is to estimate the Hölder exponent that measures the regularity of the stochastic process. Finally, we report an empirical analysis of the stock market and provide a brief economic interpretation.

This paper proceeds as follows. Section 2, 3, and 4 discuss the definitions and properties of the Bm, fBm, and mBm. Section 5 presents the main theoretical frameworks underpinning this paper and estimate the Hölder exponent. Section 6 concludes the paper. The appendix provides MATLAB code. In the next section, we introduce the definitions and properties of the Brownian motion.

2 Brownian Motion (Bm)

In this section, we recall the definitions and properties of the Bm, fBm, and mBm. All the stochastic processes in this paper are defined on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. Before we introduce the definitions, one should recall that the properties of a Gaussian process are completely determined by its expectation $E(Z(t))$ and covariance $cov(Z(s), Z(t))$, where $s, t \in \mathbb{R}$ and $Z(.)$ is a random process. In physics, Brownian motion describes the random movement of a particle suspended in fluid whose continuous path is influenced by the pressure applied by the micro particles. This motion, modeled mathematically by a stochastic process, is also known as Wiener process. To a large extent, the importance of Wiener process is explained by its fundamental role in stochastic calculus and in the limit theorems for random processes (invariance principle) (Lifshits, 2012).

2.1 The definitions of Brownian motion

A real-valued stochastic process $\{B(t)\}_{t\in[0,+\infty)}$ is called a Brownian motion if and only if the following properties hold:

(1) The Bm starts at zero with the initial value 0:

$$B(0) = 0.$$

(2) The expected value is given by:

$$E(B(t)) = 0.$$

(3) The variance is given by:

$$Var(B(t)) = E(B(t)^2) = t.$$

(4) The covariance kernel of this centered Gaussian process is given by: Assume that $t \geq s$, and for any $s, t \geq 0$,

$$Cov(B(s), B(t)) = \frac{1}{2}(|t| + |s| - |t-s|) = min\{s, t\}.$$

2.2 The properties of Brownian motion

In this subsection, we discuss some basic properties of standard Brownian motion (Bm). Brownian motion, a real-valued stochastic process $\{B(t)\}_{t\in[0,+\infty)}$, has the following properties:

(1) Hölder Continuity: A sample path of Bm is almost surely Hölder continuous: A function $B : \mathbb{R} \to \mathbb{R}$ is a H*ölder* function of order $1/2 - \xi$ for small enough $\xi > 0$.

$$\exists C > 0, |B(t) - B(s)| \approx C|t-s|^{1/2-\xi}, \forall s, t \in \mathbb{R}.$$

(2) Non-differentiability: A sample path of Brownian motion is nowhere differentiable.

$$\lim_{\varepsilon\to 0+} \frac{|B(t+\varepsilon) - B(t)|}{\varepsilon} = +\infty.$$

(3) Stationary and independent increments: For any $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1}),$$

are mutually independent [18]. In addition, Brownian motion is not a long-range dependent stationary process (no long-range dependence).

(4) Self-similarity: Brownian motion exhibits self-similarity. For any $c > 0$, the process $\left\{Y(t) := \frac{W(ct)}{\sqrt{c}}\right\}_{t\in\mathbb{R}}$ is also a Brownian motion.

3 Fractional Brownian Motion (fBm)

The fractional Brownian motion was first introduced within a Hilbert space framework by Andrei Kolmogorov in 1940 (Biagini, Hu, Øksendal, and Zhang, 2008). Then the name of fractional Brownian motion is due to B. Mandelbrot and J. Van Ness who provided a stochastic integral representation of this process in terms of a standard Brownian motion (Mandelbrot and Van Ness, 1968). The fBm has turned out to be a powerful tool in modeling and has been applied in many areas, such as hydrology, finance, signals and images processing, and telecommunications. Fractional Brownian motion, a typical example of self-similar process whose increments exhibit self-similarity exponent $H \in (0, 1)$, is a real centered Gaussian process with stationary increments $\{B^H(t)\}_{t\in\mathbb{R}}$ with covariance function $cov(B^H(t), B^H(s))$.

3.1 The definitions of fractional Brownian motion

A fractional Brownian motion (fBm) $\{B^H(t)\}_{t\in\mathbb{R}}$ of Hurst index $H \in (0, 1)$ is a continuous and centered Gaussian process with stationary increments $\{B^H(t)\}_{t\in\mathbb{R}}$ with covariance function $cov(B^H(t), B^H(s))$.

(1) The fBm starts at zero with the initial value 0:

$$B^H(0) = 0.$$

(2) The expected value is given by:

$$E(B^H(t)) = 0.$$

(3) The variance is given by:

$$Var(B^H(t)) = E(B^H(t)^2) = |t|^{2H}.$$

(4) The covariance kernel of this centered Gaussian process is given, for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$, by:

$$Cov(B^H(s), B^H(t)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Note that when H =1/2, fBm is an extension of the well known Brownian motion.

(5) Here we discuss the two representations of fractional Brownian motion (Bertrand, Hamdouni, and Khadhraoui, 2010).

- Harmonizable representation of fBm: For every $t \in \mathbb{R}$,

$$B^H(t) = \int_{\mathbb{R}} \frac{(e^{its} - 1)}{|s|^{H+1/2}} \mathrm{d}\hat{B}(s), \tag{1}$$

where the random measure $\mathrm{d}\hat{B}(s)$ satisfies

$$\mathrm{d}\hat{B}(s) = \mathrm{d}B_1(s) + \mathrm{i}\mathrm{d}B_2(s), \tag{2}$$

$\mathrm{d}B_1(s)$ and $\mathrm{d}B_2(s)$ being two independent real-valued Brownian measures (Ayache and Véhel, 2004).

- Non-anticipative moving average representation of fBm:

$$B^H(t) = \frac{1}{\Gamma(H+1/2)} \left(\int_{-\infty}^{0} \left((t-s)^{H-1/2} - (-s)^{H-1/2} \right) \mathrm{d}B(s) + \int_0^t (t-s)^{H-1/2} \mathrm{d}B(s) \right), \tag{3}$$

where $B(s)$ is a standard Brownian motion and Γ represents the gamma function, is a fBm with Hurst index $H \in (0, 1)$ (Peltier and Lévy Véhel, 1995).

In this paper, however, we only consider the moving average representation of fBm as:

$$B^H(t) = \int_0^t (t-s)^{H-1/2} \mathrm{d}B(s), \tag{4}$$

where $B(s)$ is a standard Brownian motion.

3.2 The properties of fractional Brownian motion

With the definition of fBm, a fractional Brownian motion (fBm) $\{B^{(H)}(t)\}_{t\in\mathbb{R}}$ of Hurst index $H \in (0, 1)$ has the following properties (Bertrand, Hamdouni, and Khadhraoui, 2010):

(1) Self-similarity: fBm exhibits self-similarity. For any $\alpha > 0$, $\{B^H(\alpha t)\}_{t\in\mathbb{R}}$ follows the same distribution as the process $\{\alpha^H B^H(t)\}_{t\in\mathbb{R}}$.

$$\forall \alpha > 0, \{B^H(\alpha t)\}_{t\in\mathbb{R}} \overset{\text{in dist'n}}{=} \{\alpha^H B^H(t)\}_{t\in\mathbb{R}}.$$

(2) Stationary increments: fBm has stationary increments, i.e., $B^H(t) - B^H(s)$ has the same law of $B^H(t-s)$ for all $s, t \geq 0$ and $t \geq s$.

(3) Hölder continuity: The sample paths of fBm $B^H(.)$ are almost surely Hölder continuous of order strictly less than H and for small enough $\xi > 0$.

$$\exists C > 0, |B^H(t) - B^H(s)| \approx C|t-s|^{H-\xi}, \forall s, t \in \mathbb{R}.$$

(4) Non-differentiability: $B^H(t)$ almost surely does not have differentiable sample paths with probability one.

$$\lim_{\varepsilon \to 0+} \frac{|B^H(t+\varepsilon) - B^H(t)|}{\varepsilon} = +\infty.$$

(5) Long-range dependence: fBm exhibits long-range dependence of its increments when $H > 1/2$.

$$\sum_{n=-\infty}^{+\infty} |R(n)| = +\infty,$$

where $R(n) = cov(X_t, X_{t+n})$.

(6) The roughness of fBm path remains everywhere the same since:

$$\mathbb{P}\left\{\forall t \in \mathbb{R} : \alpha_{B^H}(t) = H\right\} = 1, \tag{5}$$

where the pointwise Hölder exponent $\{\alpha_{B^H}(t)\}_{t\geq 0}$ of fBm denotes

$$\alpha_{B^H}(t) = sup\left\{\alpha, \lim_{h\to 0} \frac{|B^H(t+h) - B^H(t)|}{|h|^\alpha} < +\infty\right\}, \tag{6}$$

which allows to measure the local variations of regularity of $\{B^H(t)\}_{t\in\mathbb{R}}$. The above relation (5) means that the pointwise Hölder exponent of fBm is not allowed to change from one time to another.

(7) Three additional features of fractional Brownian motion (Bertrand, 2005):

• Antipersistent behavior (short-range dependence):
If $0 < H < 1/2$, $B^H(t)$ shows a more irregular local behavior (implying H close to 0).

• Mean reversion behavior:
If $H = 1/2$, $B^H(t)$ reduces to a standard Brownian motion which has independent increments property (i.e., $R(n) = 0$, $\forall n \neq 0$).

• Persistent behavior (long-range dependence):
If $1/2 < H < 1$, $B^H(t)$ shows a smoother local behavior (implying H close to 1).

Figure 1 and Figure 2 respectively show the simulation of fBm paths with two different Hurst exponent values ($H = 0.25$ and $H = 0.75$). One can easily find that the greater the Hurst exponent value, the smoother the trajectory of fBm and vice versa. However, fBm has its drawback due to the constant Hurst parameter everywhere in path. That is, since H is independent on t, the regularity of the fBm is the same along its path. fBm thus is not adapted to model processes which display both features of a very irregular local behavior and long-range dependence at the same time (Ayache, Cohen, and Lévy Véhel, 2000). To overcome this problem, multifractional Brownian motion has been introduced independently by Peltier and Lévy Véhel (Peltier and Lévy Véhel, 1995), and Benassi (Benassi, Jaffard, and Roux, 1997). In the next subsection, we recall the definitions and properties of mBm.

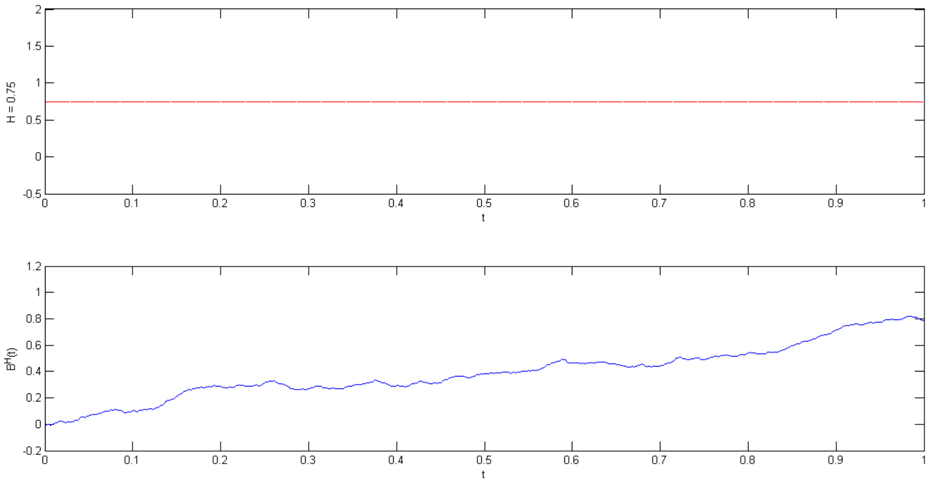

Figure 1: Simulation of fBm paths with $H = 0.75$.

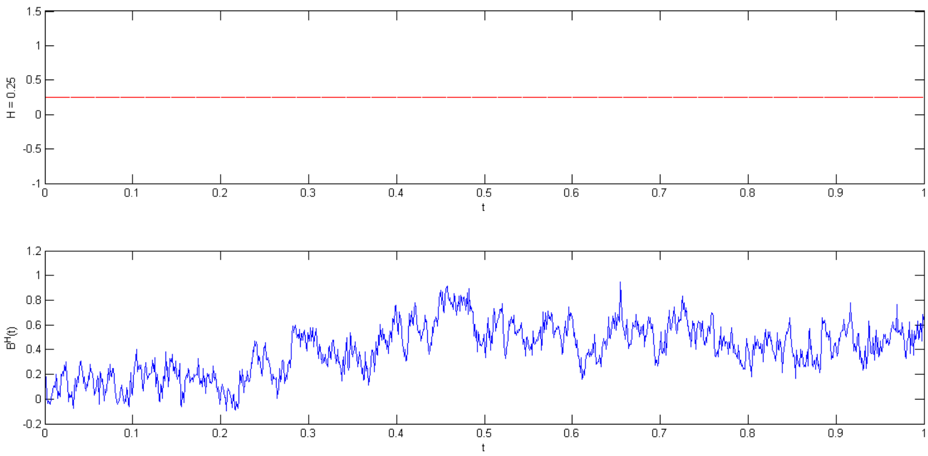

Figure 2: Simulation of fBm paths with $H = 0.25$.

4 Multifractional Brownian Motion (mBm)

Multifractional Brownian motion (mBm) is a non-stationary increments Gaussian process that generalizes the well-known fractional Brownian motion by allowing the process pointwise regularity to change over time, even very erratically (Bianchi, Pantanella, and Pianese, 2013). The mBm has been developed to overcome the limitations of fractional Brownian motion (fBm). Roughly speaking, mBm is obtained by replacing the Hurst parameter H of fBm by a smooth function $t \mapsto H(t)$.

4.1 The pointwise Hölder exponent

The pointwise Hölder exponent provides a measure of the local Hölder regularity of a process path in neighborhood of some fixed point t. Similarly to fBm the pointwise

Hölder regularity of mBm can be prescribed with its functional parameter. Thus the pointwise Hölder exponent of the stochastic process $\{X(t)\}_{t\geq 0}$ is at the point t defined as the following (Ayache and Lévy Véhel, 2004):

$$\alpha_X(t) = sup\left\{\alpha, \lim_{h\to 0}\frac{|X(t+h)-X(t)|}{|h|^{\alpha}} < +\infty\right\}, \tag{7}$$

which allows to measure the local variations of regularity of $\{X(t)\}_{t\geq 0}$.

4.2 The definitions of multifractional Brownian motion

Let $H(t)$ be a H$\ddot{o}lder$-continuous function in the interval $t \in [0,1]$ with H$\ddot{o}lder$ exponent $\beta > 0$, such that for any $t > 0$ we have $0 < H(t) < min(1,\beta)$ (Costa and Vasconcelos, 2003). Then the multifractional Brownian motion with functional parameter $H(.)$ is the continuous Gaussian process $\{X(t)\}_{t\geq 0}$ defined for every t:

(1) The mBm starts at zero with the initial value 0:

$$X(0) = 0.$$

(2) The expected value is given by:

$$E(X(t)) = 0.$$

(3) The covariance kernel of this centered Gaussian process is given, for all $s,t \in \mathbb{R}$, by:

$$Cov(X(s),X(t)) = \tfrac{1}{2}(|t|^{H(s)+H(t)} + |s|^{H(s)+H(t)} - |t-s|^{H(s)+H(t)}).$$

Note that when $H(t) = H$, $X(t) = B^H(t)$, which implies that mBm reduces to fBm with parameter H. Thus, multifractional Brownian motion is an extension of fractional Brownian motion.

(4) Here we discuss the two representations of multifractional Brownian motion.

- Harmonizable representation of mBm: For every $t \in \mathbb{R}$,

$$X(t) = B^{H(t)}(t) = \int_{\mathbb{R}} \frac{\left(e^{its}-1\right)}{|s|^{H(t)+1/2}} \mathrm{d}\hat{B}(s). \tag{8}$$

where the random measure $\mathrm{d}\hat{B}(s)$ satisfies

$$\mathrm{d}\hat{B}(s) = \mathrm{d}B_1(s) + \mathrm{i}\mathrm{d}B_2(s), \tag{9}$$

$\mathrm{d}B_1(s)$ and $\mathrm{d}B_2(s)$ being two independent real-valued Brownian measures (Ayache and Lévy Véhel, 2004).

- Non-anticipative moving average representation of mBm:

$$X(t) = B^{H(t)}(t) = \frac{1}{\Gamma(H+1/2)}\int_{\mathbb{R}}\left((t-s)_+^{H(t)-1/2} - (-s)_+^{H(t)-1/2}\right)\mathrm{d}B(s), \tag{10}$$

where $\mathit{\Gamma}(.)$ is the gamma function and $B(s)$ denotes the standard Brownian motion.

In this paper, we only consider the moving average representation of mBm as:

$$X(t) = B^{H(t)}(t) = \int_0^t (t-s)^{H(t)-1/2}\,\mathrm{d}B(s), \tag{11}$$

where $B(s)$ is a standard Brownian motion.

4.3 The properties of multifractional Brownian motion

With the definition of mBm, a multifractional Brownian motion (mBm) $\{X(t)\}_{t\geq 0}$ of Hurst index $H(t)$ has the following properties:

(1) Non-stationary increments: $X(t)$ has non-stationary increments since it can be shown as:

$$E\left[(X(t+s)-X(s))^2\right] \approx t^{2H(t)}.$$

Note that because of its non-stationarity, mBm is no longer a self-similar process either. However, we can define the concept of locally asymptotically self-similarity.

(2) Locally asymptotically self-similarity: At any point t, mBm is locally asymptotically self-similar with the index $H(t)$, more precisely,

$$\forall \alpha > 0, \left\{B^{H(t)}(\alpha t)\right\}_{t\in\mathbb{R}} \overset{\text{in dist'n}}{\approx} \left\{\alpha^{H(t)} B^{H(t)}(t)\right\}_{t\in\mathbb{R}}.$$

(3) Hölder continuity: The sample paths of mBm $B^{H(t)}(.)$ are almost surely Hölder continuous of order strictly less than $H(t)$ and for small enough $\xi > 0$.

$$\exists C > 0, |X(t) - X(s)| \approx C|t-s|^{H(t)-\xi}, \forall s, t \in \mathbb{R}.$$

(4) Nondifferentiablity: $X(t)$ almost surely does not have differentiable sample paths with probability one.

$$\lim_{\varepsilon \to 0+} \frac{|X(t+\varepsilon) - X(t)|}{\varepsilon} = +\infty.$$

(5) The roughness of mBm path does not remain everywhere the same since:

$$\mathbb{P}\left\{\forall t \in [0, +\infty] : \alpha_X(t) = H(t)\right\} = 1,$$

which means that the pointwise Hölder exponent of $\{X(t)\}_{t\geq 0}$ may exhibit very irregular behavior.

5 Empirical Analysis

5.1 Data

To implement the multifractional Brownian motion for our empirical analysis, the data have to be correlated. Since we find that a big event such as financial crisis would cause huge market fluctuations (volatilities), thus, in this paper we only look at the data of post–2008 financial crisis periods. In addition, we need data, which are not dominated by some trends, to use the exponential smoothing regression. The data consists of 1518 daily values corresponding to the sample periods (from March 2009 to June 2015). As the case for the financial analysis, we deal with the logarithm of the Dow Jones Industrial Average plotted in Figure 3.

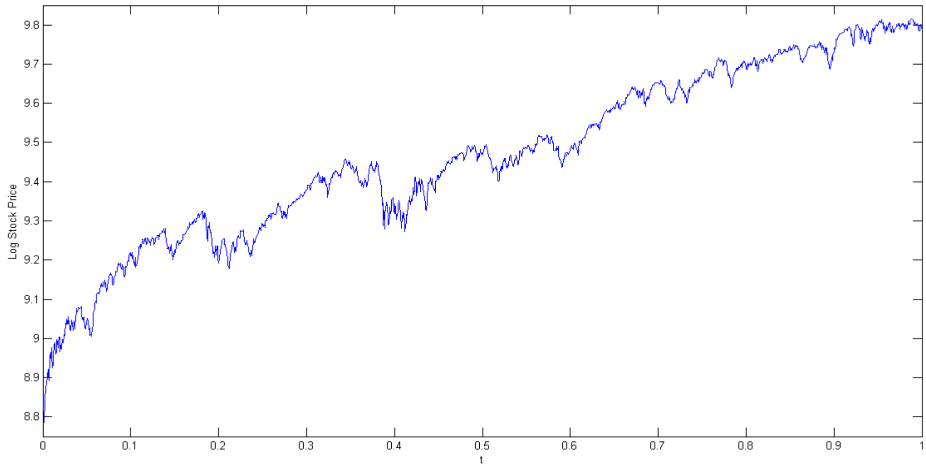

Figure 3: Daily log stock price.

5.2 Model

Since prices are always nonnegative, they can be described as exponential of the sum of a regular process and a stochastic process, and price processes are better fitted by a log normal process (Bertrand, Hamdouni, and Khadhraoui, 2010). Thus the model is given by:

$$Z(t) = exp\,[f(t) + cX(t)], \tag{12}$$

where
$f(t)$: deterministic function,
$c = \sqrt{var(logZ(t) - f(t))}$: scaling factor,
$X(t) = (logZ(t) - f(t))/c$: standard mBm.

Then our model of mBm with the functional parameter $H(t)$ is given by:

$$X(t) = \frac{logZ(t) - f(t)}{\sqrt{var(logZ(t) - f(t))}}. \tag{13}$$

5.3 Algorithm

We first implement the exponential smoothing regression to derive the best fitted result of linear regression on $f(t)$. Then we estimate $X(t)$ and c. In addition, we compute the Hurst index of $X(t)$ by using the pointwise Hölder exponent estimator given by Fraclab. Fraclab is a signal and image processing toolbox based on fractal and multifractional methods which has been developed at INRIA by Lévy Véhel and his former students. Lastly, to develop the algorithms, we will take the following four steps:

- Step 1: Estimating $f(t)$.
  Suppose $logZ(t) = f(t) + \epsilon$. Then we first estimate $f(t)$ by the exponential smoothing regression. Figure 4 shows the log stock price and estimated log stock price, $\hat{f}$. Also, we show the log stock price and estimated log stock price, $\hat{f}$ by splitting the time span into two segments. Figure 5 and Figure 6 respectively

show two different time segments.

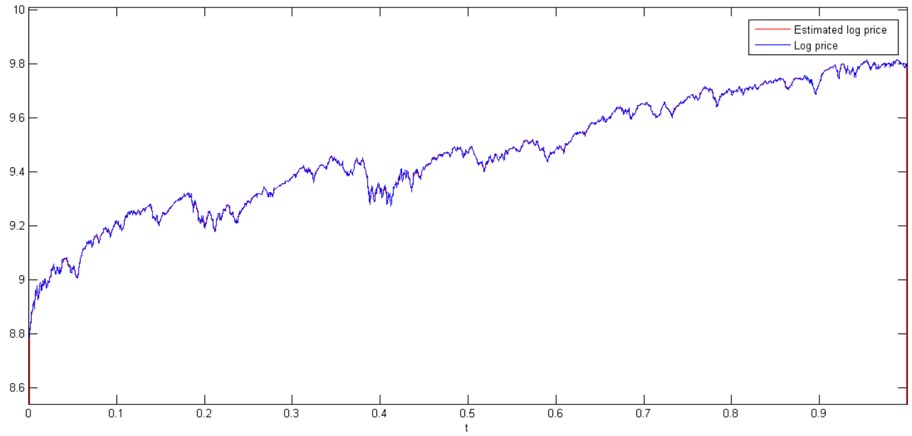

Figure 4: Log price and estimated log price, $\hat{f}$, by exponential smoothing regression.

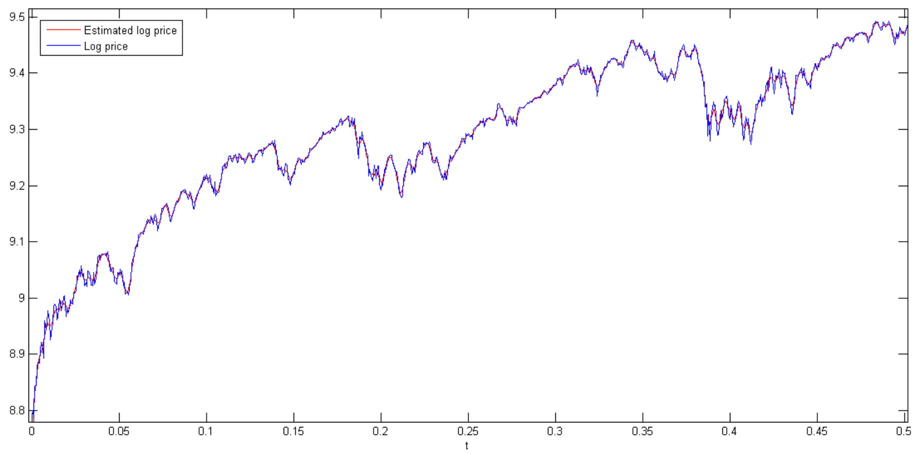

Figure 5: Log price and estimated log price, $\hat{f}$, for $0 \leq t \leq 0.5$.

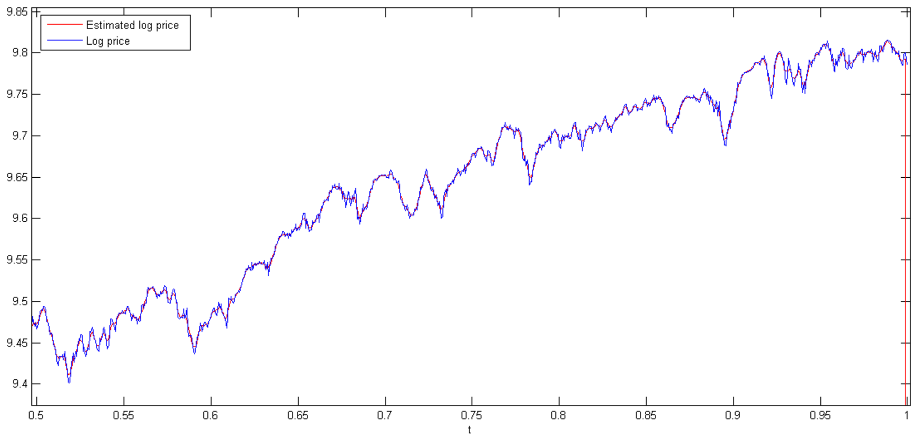

Figure 6: Log price and estimated log price, $\hat{f}$, for $0.5 \leq t \leq 1$.

- Step 2: Estimating $X(t)$ and c.
  Since $X(t)$ can be written as:

$$X(t) = \frac{logZ(t) - f(t)}{c}. \tag{14}$$

  Thus, $X(t)$ can be estimated as:

$$\hat{X}(t) = \frac{logZ(t) - \hat{f}(t)}{\sqrt{var(logZ(t) - \hat{f}(t))}}. \tag{15}$$

  Also, since $c = \sqrt{var(logZ(t) - f(t))}$, we can estimate c as:

$$\hat{c} = \sqrt{var(logZ(t) - \hat{f}(t))}. \tag{16}$$

- Step 3: Estimating $H(t)$ by using $\hat{X}(t)$ starting from $logZ(t) - \hat{f}(t)$.
  We estimate the Hölder function of the multifractional Brownian motion using the generalized quadratic variations (GQVs) given by Fraclab. Figure 7 shows the estimated Hurst parameter, $\hat{H}(t)$, which is also a stochastic process. In addition, with the estimated Hurst parameter, $\hat{H}(t)$, we generate a multifractional Brownian motion (mBm) using enhanced Wood and Chan synthesis method give by Fraclab. Figure 8 shows estimated multifractional Brownian motion, $\hat{X}(t)$, and generated multifractional Brownian motion, $X(t)$.

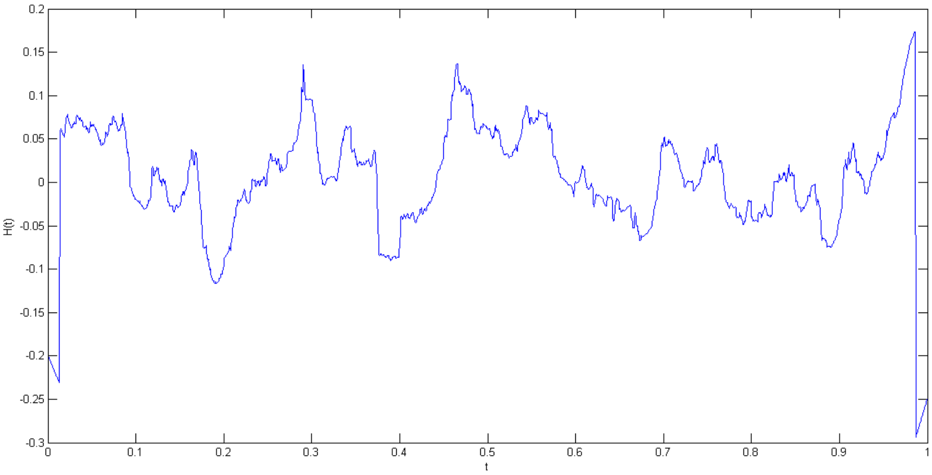

Figure 7: Estimated Hurst parameter, $\hat{H}(t)$, by GQVs.

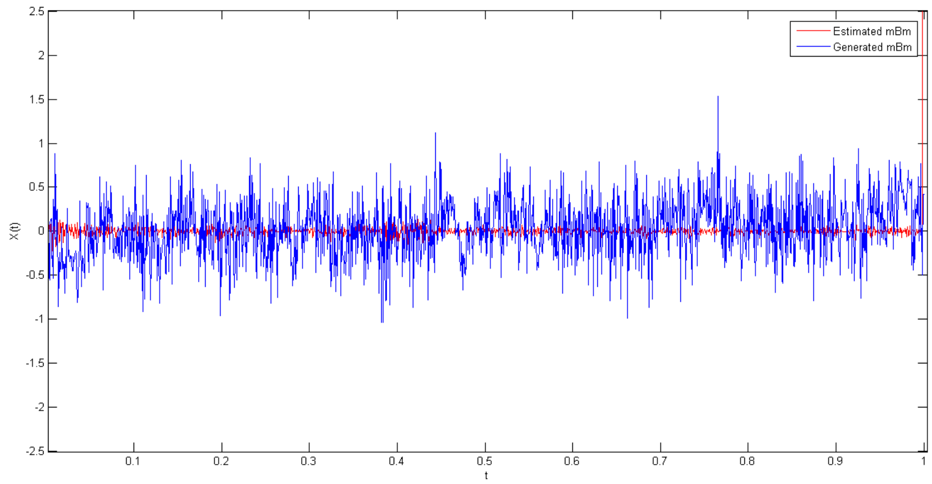

Figure 8: Generated mBm $X(t)$ and estimated mBm $\hat{X}(t)$.

- Step 4: Predicting $Z(t) = exp\,[f(t) + cX(t)]$.
  Finally, we can estimate $Z(t)$. Thus the estimated stock price (DJI) is given by:

$$\hat{Z(t)} = exp\left[\hat{f(t)} + \hat{X}(t)\sqrt{var(logZ(t) - \hat{f}(t))}\right]. \tag{17}$$

As Figure 9 shows, our mBm model well estimated the dynamics of the daily stock price (DJI). Also, we plot it by splitting into the two time segments in Figure 10 and Figure 11.

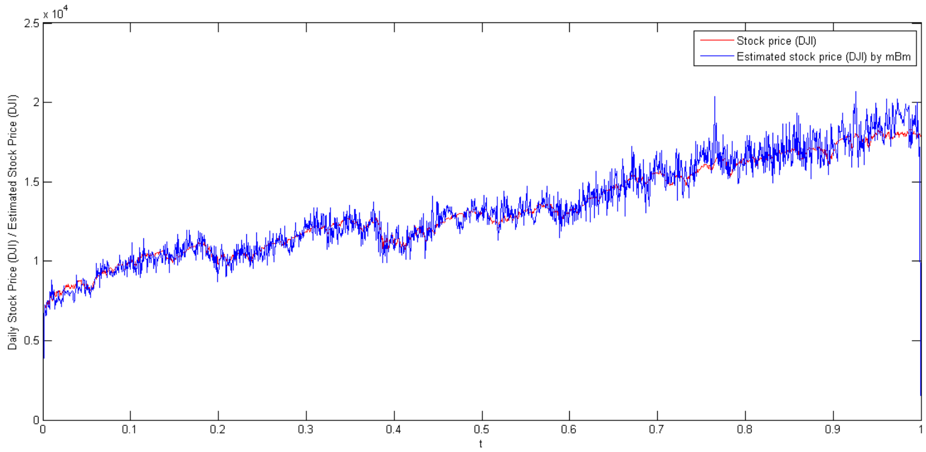

Figure 9: Stock price (DJI) and estimated stock price (DJI) by mBm

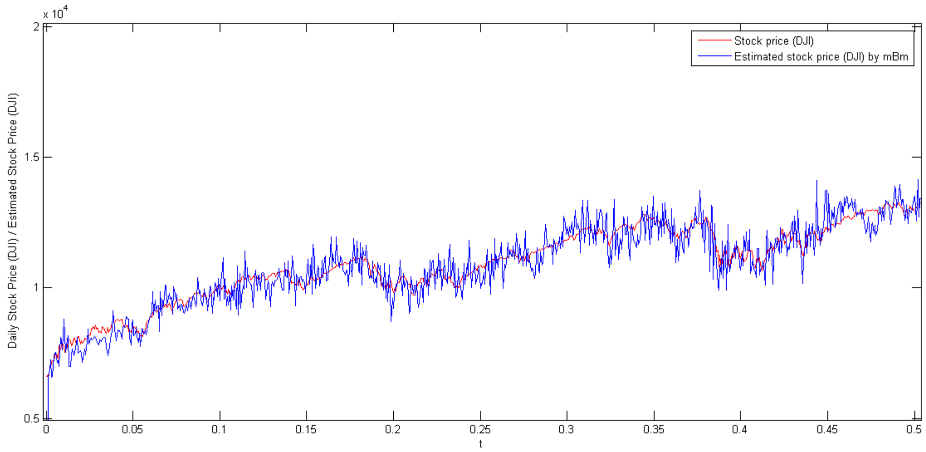

Figure 10: Stock price (DJI) and estimated stock price (DJI) by mBm for $0 \leq t \leq 0.5$

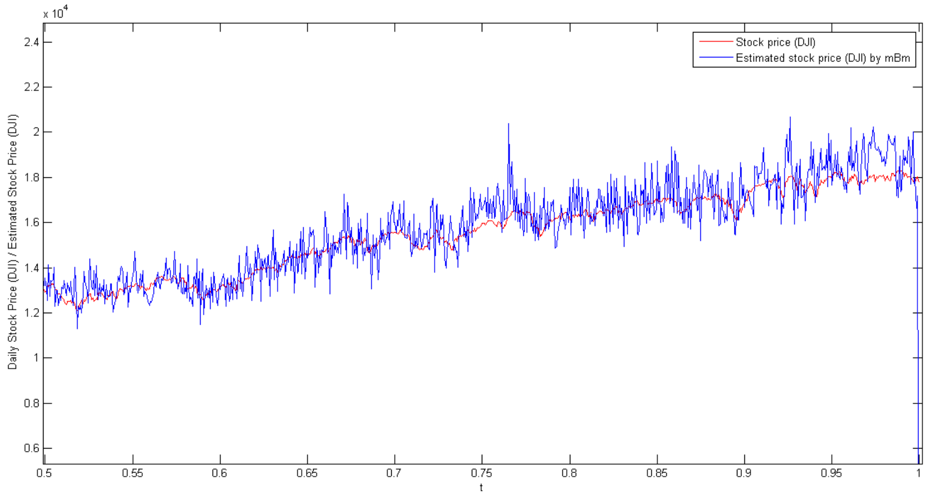

Figure 11: Stock price (DJI) and estimated stock price (DJI) by mBm for $0.5 \leq t \leq 1$

Figure 12 shows the trajectories of the stock price (DJI) and estimated Hurst parameter. The estimated pointwise Hölder exponent seems to move erratically as we expect. In addition, we find that the estimated Hölder index ranges between -0.2933 and 0.1736. One should note that a stochastic process shows a very smooth (differentiable) path when $H(t) > 1$, and it even shows sudden jumps in the path when $H(t) < 0$. In the following sections, we will briefly discuss limitations of the research and conlude the paper.

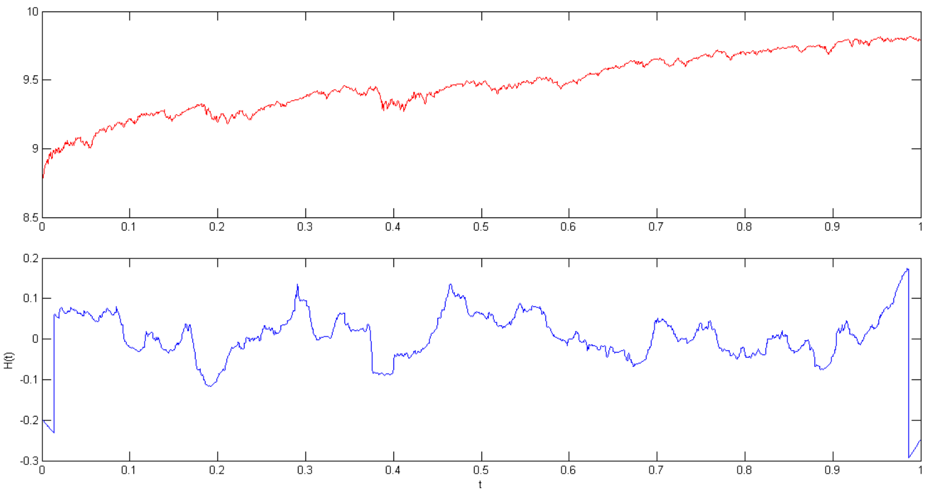

Figure 12: Stock price (DJI) and estimated Hurst parameter $\hat{H}(t)$

6 Limitations of the Research

Our analysis does encounter some limitations. For simplicity of the presentation, we mainly focus on the application of mBm as a model of financial dynamics in estimating a stock market index, DJI. Our analysis can be extended to understand price dynamics of other asset classes such as bonds, commodities, or cryptocurrencies. Our empirical analysis is done on the short sample period. It is of great interest to see what can be further investigated with an extended sample period even including the 2008 financial market crisis period. Lastly, our study does not explicitly provide an interesting correlation between the asset prices and Hurst parameter. That is, the empirical examination on the relationship between the predictability of financial time series and Hurst parameter can be sought. Hurst parameter is known as a measure for the behavior of the market. Recall that it does show if the market behaves in a random ($0 < H < 1/2$), trending ($1/2 < H < 1$), or mean-reversion ($H = 1/2$) manner. In this regard, the further applications of mBm can be studied to find empirical evidence of a correlation between the asset prices and Hurst parameter.

7 Conclusion and Future Research

In this paper, we focus on a new model of the stock price with the multifractional Brownian motion. Also, we show that a process like mBm can be used to simultaneously capture the fluctuations of the asset price dynamics and the long-range dependence of financial time series. Thus, we argue that with a proper functional parameter $H(t)$ we can generate mBm that can reproduce the stylized facts characterizing financial time series. In addition, the path created by $H(t)$ can describe the state of the market. For example, it can characterize both 'bullish' and 'bearish' markets and the frenetic buy-and-sell behaviors affecting the markets during the financial crises. In this light, the model can potentially be applied to explain investors' diverse trading behaviors, especially their reactions to both bubbles and crashes in the financial market. We will also be able to explain how the behavior (trend) of the Hurst parameter, $H(t)$, can capture the complex financial dynamics of a big market event such as the financial crisis. We leave to future research the detailed analysis of the model prediction with mBm since we have achieved the model estimation analysis in this paper.

Appendix

Z = Close(1 : 1581)
LZ = log(Z)
LZ =LZ’
t = linspace$(0, 1$, length(LZ))
f = smoothts(LZ)
plot(t, f, ‘r’)
line(t, LZ)
X = (LZ $-f$) / $[(var((LZ) - f)]^{\frac{1}{2}}$
G = *exp*(linspace(1, log(100), length(LZ)))’
[H,G] = estimGQV1DH(G.*X’, 0.6, 1, 1, 5)
mBm0 = mBmQuantifKrigeage(length(LZ), 10, H, 1, 1)
plot(t, X, ‘r’)
line(t, mBm0)
EZ = *exp*(f + sqrt(var(LZ-f))*mBm0’)
plot(t, Z, ‘r’)
line(t, EZ)

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